SELF-INDUCED BANACH ALGEBRAS

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Abstract. A Banach algebra $\mathfrak{A}$ is self-induced if the multiplication $\mathfrak{A}\underset{\mathfrak{A}}{\widehat{\otimes}}\mathfrak{A} \mapsto \mathfrak{A}$ is an isomorphism. The class of self-induced Banach algebras is a natural generalization of unital Banach algebras, providing a fertile framework for developing homological aspects of unital Banach algebras. Elementary results with applictions to computations of the bounded cohomology groups $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$, with emphasis on $\mathfrak{A} = \mathcal{A}(X)$, the approximable operators on a Banach space X, are given.

0. Introduction. A Banach algebra $\mathfrak{A}$ is *self-induced* if the multiplication

$$\mathfrak{A}\underset{\mathfrak{A}}{\widehat{\otimes}}\mathfrak{A} \xrightarrow{\mu_{\mathfrak{A}}} \mathfrak{A}$$

is an isomorphism. The concept of self-induced Banach algebras was introduced in [G2] with the purpose of providing a setting for a Banach algebra theory of Morita equivalence rich enough to include commonly occurring Banach algebras. Self-inducedness is in liking with *H-unitality*, the concept introduced by M. Wodzicki in [W]. The Banach algebraic version of H-unitality is stipulated in the requirement that the bar complex of $\mathfrak{A}$

$$0 \longleftarrow \mathfrak{A} \longleftarrow \mathfrak{A}\widehat{\otimes}\mathfrak{A} \longleftarrow \mathfrak{A}\widehat{\otimes}\mathfrak{A}\widehat{\otimes}\mathfrak{A} \longleftarrow \cdots$$

is pure-exact, or equivalently, the bounded homology groups $\mathcal{H}_n(\mathfrak{A}, \mathfrak{X}) = \{0\},\ n \geq 1$ for any annihilator Banach $\mathfrak{A}$-bimodule $\mathfrak{X}$. It is rather elementary that if $\mathfrak{A}$ is H-unital as a Banach algebra, then the inclusion $\mathfrak{A} \mapsto \mathfrak{A}_+$, where $\mathfrak{A}_+$ is the unitization of $\mathfrak{A}$, implements an isomorphism $\mathcal{H}^n(\mathfrak{A}_+, \mathfrak{A}_+^*) \cong \mathcal{H}^n(\mathfrak{A}, \mathfrak{A}^*)$. Rephrasing this, the admissable short exact sequence $0 \to \mathfrak{A} \to \mathfrak{A}_+ \to \mathbb{C} \to 0$ implements a long exact sequence $\cdots \to \mathcal{H}^n(\mathbb{C}, \mathbb{C}^*) \to \mathcal{H}^n(\mathfrak{A}_+, \mathfrak{A}_+^*) \to \mathcal{H}^n(\mathfrak{A}, \mathfrak{A}^*) \to \cdots$. However, the main result of [W] that this excission property holds in general, is far from elementary: If $\mathfrak{A}$ is H-unital as a Banach algebra, then any weakly admissable extension $0 \to \mathfrak{A} \to \mathfrak{B} \to \mathfrak{C} \to 0$ implements a long exact sequence

$$\cdots \to \mathcal{H}^n(\mathfrak{C}, \mathfrak{C}^*) \to \mathcal{H}^n(\mathfrak{B}, \mathfrak{B}^*) \to \mathcal{H}^n(\mathfrak{A}, \mathfrak{A}^*) \to \cdots .$$

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Being self-induced may be seen as H-unitality in "degree $1+\frac{1}{2}$". More precisely $\mathfrak{A}$ is self-induced is equivalent to each of

(1) $\mathcal{H}^1(\mathfrak{A},\mathfrak{X})=\mathcal{H}^2(\mathfrak{A},\mathfrak{X})=\{0\}$ for any annihilator bi-module $\mathfrak{X}$.
(2) $\mathcal{H}^0_{\mathrm{bar}}(\mathfrak{A})=\mathcal{H}^1_{\mathrm{bar}}(\mathfrak{A})=\{0\}$, where $\mathcal{H}^\bullet_{\mathrm{bar}}$ is the cohomology of the dual bar-complex.

Besides providing the framework for a Morita theory, self-induced Banach algebras share a basic cohomological property with unital algebras: If $\mathfrak{A}\to\mathfrak{B}$ is an embedding of a self-induced Banach algebra $\mathfrak{A}$ as an ideal in $\mathfrak{B}$, then restriction implements isomorphisms $\mathcal{H}^1(\mathfrak{A},\mathfrak{B}^*)\cong H^1(\mathfrak{B},\mathfrak{A}^*)\cong H^1(\mathfrak{A},\mathfrak{A}^*)$, see [G3, Lemma 2.2]. In the present paper we show some elementary properties of self-induced Banach algebras with the aim of computing the cohomology groups $\mathcal{H}^1(\mathfrak{A},\mathfrak{A}^*)$. An imporpant aspect will be Morita invariance of Hochschild cohomology for self-induced Banach algebras. We shall apply our findings to algebras of the type $\mathcal{A}(X)$, the approximable operators on a Banach space X, thereby improving on previous results, in particular concerning direct sums and tensor products, of [B1], [B2], and [G3].

1. Preliminaries. For Banach spaces X and Y we consider the following spaces of operators

$$\begin{aligned}\mathcal{F}(X,Y)&=\{\text{finite rank operators } X\to Y\}\\ \mathcal{A}(X,Y)&=\{\text{approximable operators } X\to Y\}=\overline{\mathcal{F}(X,Y)}\\ \mathcal{I}(X,Y)&=\{\text{integral operators } X\to Y\}\\ \mathcal{B}(X,Y)&=\{\text{bounded operators } X\to Y\}\end{aligned}$$

As customary we shall write Operators(X) for Operators(X,X).

We shall use $\langle\cdot\,,\cdot\rangle$ to denote the bilinear form $\langle x,x^*\rangle=x^*(x)$ for $x\in X, x^*\in X^*$. The corresponding bilinear functional $\mathrm{tr}\colon X\otimes X^*\to\mathbb{C}$ is *the canonical trace*. Since it *is* canonical it is not annotated with the space X.

The definitions of Banach (co)homological concepts are standard and can be found for example in [H] and [J]. For Banach algbras $\mathfrak{A}$ and $\mathfrak{B}$ we use the notations $\mathfrak{A}\mathbf{mod}$, $\mathbf{mod}\mathfrak{A}$, and $\mathfrak{A}\mathbf{mod}\mathfrak{B}$ for the categories of Banach left, Banach right $\mathfrak{A}$-modules, and Banach $\mathfrak{A}$-$\mathfrak{B}$-bimodules. The unitization of $\mathfrak{A}$ is the Banach algebra given by the augmentation $\mathfrak{A}\to\mathfrak{A}_+\to\mathbb{C}$. We denote bounded Hochschild homology and cohomology for $\mathfrak{A}$ with coefficients in $\mathfrak{X}\in\mathfrak{A}\mathbf{mod}\mathfrak{A}$ by $\mathcal{H}_n(\mathfrak{A},\mathfrak{X})$ and $\mathcal{H}^n(\mathfrak{A},\mathfrak{X})$. We shall almost exclusively be interested in $n=1$.

Recall

1.1 Definition. Let $\mathfrak{A}$ be a Banach algebra, let $P\in\mathbf{mod}\mathfrak{A}$ and $Q\in\mathfrak{A}\mathbf{mod}$. We define

$$P\underset{\mathfrak{A}}{\widehat{\otimes}}Q=P\widehat{\otimes}Q/N$$

where $\widehat{\otimes}$ is the projective tensor product and $N=\operatorname{clspan}\{p.a\otimes q-p\otimes a.q\mid p\in P,q\in Q,a\in\mathfrak{A}\}$. Thus, $P\underset{\mathfrak{A}}{\widehat{\otimes}}Q$ is the universal object for linearizing bounded, $\mathfrak{A}$-balanced bilinear maps $P\times Q\to Z$ into Banach spaces Z.

For $P\in\mathfrak{A}\mathbf{mod}$ the we denote the linear map implemented by multiplication by $\mu_P\colon\mathfrak{A}\underset{\mathfrak{A}}{\widehat{\otimes}}P\to P$ and likewise for right and two-sided modules. If it is unambiguous we shall omit the subscript.

All (multi)-linear maps will be assumed to be bounded, and all occuring series in Banach spaces are assumed to be absolutely convergent.

2. Morita contexts and derivations of self-induced Banach algebras.

2.1 Definition. A Banach algebra $\mathfrak{A}$ is called *self-induced* if multiplication implements a bimodule isomorphism

$$\mathfrak{A}\underset{\mathfrak{A}}{\widehat{\otimes}}\mathfrak{A} \xrightarrow{\mu} \mathfrak{A}.$$

More generally, $P \in \mathfrak{A}\mathbf{mod}$ is an $\mathfrak{A}$-*induced module* if μ_P is an isomorphism, and similarly for $P \in \mathbf{mod}\mathfrak{A}$.

As mentioned, one motivation for this definition is that if $\mathfrak{A}$ is self-induced, then $\mathcal{H}^1(\mathfrak{A}_+, \mathfrak{A}_+^*) \cong \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$. However, self-inducedness is not necessary for this. More precisely we have

2.2 Proposition. *The following three statements are equivalent.*
(i) $\mathcal{H}^1(\mathfrak{A}_+, \mathfrak{A}_+^*) \cong \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$*;*
(ii) $\mathcal{H}^1(\mathfrak{A}, \mathbb{C}) = \{0\}$ *and* $\mathcal{H}^2(\mathfrak{A}, \mathbb{C}) \to \mathcal{H}^2(\mathfrak{A}, \mathfrak{A}_+^*)$ *is injective for the annihilator module* $\mathbb{C}$*;*
(iii) $\mathrm{cl}(\mathfrak{A}^2) = \mathfrak{A}$ *and for each derivation* $D: \mathfrak{A} \to \mathfrak{A}^*$ *the bilinear map* $(a, b) \mapsto \langle a, D(b)\rangle + \langle b, D(a)\rangle$ *is* $\mathfrak{A}$*-balanced.*

Proof. This is proved by inspection of the concepts involved.

2.3 Examples. (a) $\mathfrak{A}$ is self-induced, if $\mathfrak{A}$ is left or right flat and $\mathrm{cl}(\mathfrak{A}^2) = \mathfrak{A}$, in particular if $\mathfrak{A}$ is biflat. Important instances occur, when $\mathfrak{A}$ has a one-sided bounded approximate identity, in which case $\mathfrak{A}$ is in fact H-unital.
(b) For a Banach space X the Banach algebra $\mathcal{A}(X)$ is self-induced,

(1) if X has the bounded approximation property, cf. (a), or more generally if $\mu_{\mathcal{A}(X)}$ is surjective and X has the approximation property, see [G3];
(2) if $X = \ell_2(E)$, when E and E^* both are of cotype 2, [Proposition 4.2, B1];
(3) if $X = E \oplus C_p$, $1 \le p \le \infty$ with E arbitrary and C_p one of the spaces defined by W.B. Johnson, [Jo].

(c) If P is one of the spaces constructed by G. Pisier in [P1], then $\mu_{\mathcal{A}(P)}$ is surjective, but not injective.
(d) If P is the space constructed by G. Pisier in [P2], then $\mu_{\mathcal{A}(P)}$ is not surjective.
(e) $\mathcal{N}(X)$ is self-induced, if and only if X has the approximation property, [G3].
(f) The augmentation ideal of the free group on two generators, $I(\mathbb{F}_2)$, is not self-induced, yet $\mathcal{H}^1(I(\mathbb{F}_2), I(\mathbb{F}_2)^*) = \{0\}$, [G&L, Corollary 3.2].

In the setting of this paper it will be convenient to have Morita equivalence described in terms of Morita contexts. For Banach algebras with trivial annihilator this is equivalent to the definition given in [G2].

2.4 Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be self-induced Banach algebras, and let $P \in \mathfrak{A}\mathbf{mod}\mathfrak{B}$ and $Q \in \mathfrak{B}\mathbf{mod}\mathfrak{A}$. Two balanced bilinear maps $\{\ ,\ \}: P \times Q \to \mathfrak{A}$ and $[\ ,\ ]: Q \times P \to \mathfrak{B}$ are called *compatible pairings* if they implement bimodule homomorphisms $P\underset{\mathfrak{B}}{\widehat{\otimes}}Q \to \mathfrak{A}$ and $Q\underset{\mathfrak{A}}{\widehat{\otimes}}P \to \mathfrak{B}$ and

$$(\S) \qquad \{p, q\}p' = p[q, p'],\ [q, p]q' = q\{p, q'\}; \quad p, p' \in P,\ q, q' \in Q.$$

These data are collected in a *Morita context* $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, P, Q, \{\ ,\}, [\ ,\ ])$. To $\mathfrak{M}$ is associated a Banach algebra $\mathcal{M}$ consisting of 2×2 matrices

$$\begin{pmatrix} a & p \\ q & b \end{pmatrix}; \quad a \in \mathfrak{A}, b \in \mathfrak{B}, p \in P, q \in Q$$

with the product defined by means of module multiplication and the compatible pairings. The Morita context is called *full* if both pairings implement bimodule isomorphisms. The Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$ are *Morita equivalent*, if there are modules $P \in \mathfrak{A}\mathbf{mod}\mathfrak{B}$, $Q \in \mathfrak{B}\mathbf{mod}\mathfrak{A}$ and compatible pairings $\{\ ,\}$, $[\ ,\ ]$ constituting a full Morita context.

The simplest case of Morita invariance of Hochshild cohomology occurs for $n \times n$-matrices. First we investigate self-inducedness:

2.5 Proposition. *Let $\mathfrak{A}$ be a Banach algebra satisfying* $\mathrm{cl}(\mathfrak{A}^2) = \mathfrak{A}$. *Then*

$$M_n(\mathfrak{A}) \underset{M_n(\mathfrak{A})}{\widehat{\otimes}} M_n(\mathfrak{A}) = M_n(\mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} \mathfrak{A}) ,$$

where M_n denotes $n \times n$ matrices.

Proof. We prove that $M_n(\mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} \mathfrak{A})$ has the universal property defining the tensor product $\underset{M_n(\mathfrak{A})}{\widehat{\otimes}}$. Let

$$\Phi \colon M_n(\mathfrak{A}) \times M_n(\mathfrak{A}) \to Z$$

be a bounded $M_n(\mathfrak{A})$-balanced bilinear map into a Banach space Z. Let E_{ij} be the elementary matrices in $M_n(\mathbb{C})$ and let $a, a', b \in \mathfrak{A}$. Then

$$\begin{aligned} \Phi(aa'E_{ij}, bE_{kl}) &= \Phi(aE_{i1}a'E_{1j}, bE_{kl}) = \Phi(aE_{i1}, a'E_{1j}bE_{kl}) \\ &= \begin{cases} 0, & j \neq k \\ \Phi(aE_{i1}, a'bE_{1l}), & j = l \end{cases} \end{aligned}$$

Since $\mathrm{cl}(\mathfrak{A}^2) = \mathfrak{A}$, we further get $\Phi(aE_{ij}, bE_{kl}) = 0$ for $j \neq k$. Let

$$\phi_{ij}(a, b) = \Phi(aE_{i1}, bE_{1j})$$

Then ϕ_{ij} is clearly $\mathfrak{A}$-balanced. Let $\iota \colon M_n(\mathfrak{A}) \times M_n(\mathfrak{A}) \to M_n(\mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} \mathfrak{A})$ be the bilinear map

$$((a_{ij}), (b_{kl})) \mapsto (\sum_k a_{ik} \underset{\mathfrak{A}}{\otimes} b_{kj})_{ij}.$$

Then ι is $M_n(\mathfrak{A})$ balanced and the above calculation shows that if we for $U = (u_{ij}) \in M_n(\mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} \mathfrak{A})$ set

$$\widetilde{\Phi}(U) = \sum_{ij} \phi_{ij}(u_{ij})$$

then $\Phi = \widetilde{\Phi} \circ \iota$. Since $\operatorname{span} \iota(M_n(\mathcal{A}) \times M_n(\mathfrak{A}))$ is dense, $\widetilde{\Phi}$ is uniquely determined.

2.6 Corollary. *$\mathfrak{A}$ is self-induced if and only if $M_n(\mathcal{A})$ is self-induced.*

We now have elementary Morita invariance of Hochschild cohomology in degree 1.

2.7 Theorem. *Let $\mathfrak{A}$ be self-inducedlet $\iota\colon \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to M_n(\mathfrak{A})\widehat{\otimes}M_n(\mathfrak{A})$ be the map given by*

$$a \mapsto \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},$$

and let $\tau\colon M_n(\mathfrak{A})\widehat{\otimes}M_n(\mathfrak{A}) \to \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be the map

$$\tau((a_{ij}) \otimes (b_{ij})) = \sum_{i,j} a_{ij} \otimes b_{j\,i}.$$

Then the implemented maps $\mathcal{H}^1(M_n(\mathfrak{A}), M_n(\mathfrak{A})^*) \xrightarrow{\iota^*} \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$ and $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) \xrightarrow{\tau^*} \mathcal{H}^1(M_n(\mathfrak{A}), M_n(\mathfrak{A})^*)$ are inverses of one another.

Proof. If $\mathfrak{A}$ is unital, this is the usual Morita invariance of Hochschild cohomology of matrices. Consider the admissable sequence

$$\{0\} \to M_n(\mathfrak{A}) \to M_n(\mathfrak{A}_+) \to M_n(\mathbb{C}) \to \{0\}$$

We have $\mathcal{H}^1(\mathfrak{A}_+, \mathbb{C}) = \mathcal{H}^2(\mathfrak{A}_+, \mathbb{C}) = \{0\}$, since $\mathfrak{A}$ is self-induced. By Morita invariance we then have $\mathcal{H}^1(M_n(\mathfrak{A}_+), M_n(\mathbb{C})) = \mathcal{H}^2(M_n(\mathfrak{A}_+), M_n(\mathbb{C})) = \{0\}$. Since $M_n(\mathfrak{A})$ is self-induced, we also have $\mathcal{H}^1((M_n(\mathfrak{A}), M_n(\mathbb{C})) = \mathcal{H}^2(M_n(\mathfrak{A}), M_n(\mathbb{C})) = \{0\}$ for the annihilator module $M_n(\mathbb{C})$, cf. the introduction.

¿From the long exact sequence of cohomology we get a commutative diagram

$$\begin{array}{ccccccc}
\{0\} & \longrightarrow & \mathcal{H}^1(M_n(\mathfrak{A}_+), M_n(\mathfrak{A}_+)^*) & \longrightarrow & \mathcal{H}^1(M_n(\mathfrak{A}_+), M_n(\mathfrak{A})^*) & \longrightarrow & \{0\} \\
 & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & \mathcal{H}^1(M_n(\mathfrak{A}), M_n(\mathfrak{A}_+)^*) & \longrightarrow & \mathcal{H}^1(M_n(\mathfrak{A}), M_n(\mathfrak{A})^*) & \longrightarrow & \{0\}
\end{array}$$

where the vertical maps are implemented by inclusions. The rows are exact and the right vertical arrow is an isomorphism by the extension property of self-induced Banach algebras ([G3,Lemma 2.2]). Hence the left vertical arrow is an isomorphism, and we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^1(M_n(\mathfrak{A}_+), M_n(\mathfrak{A}_+)^*) & \xrightarrow[\cong]{} & \mathcal{H}^1(M_n(\mathfrak{A}), M_n(\mathfrak{A})^*) \\
\uparrow \iota^* & & \uparrow \iota^* \\
\mathcal{H}^1(\mathfrak{A}_+, \mathfrak{A}_+^*) & \xrightarrow[\cong]{} & \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)
\end{array}$$

It follows than $\iota^*\colon \mathcal{H}^1(M_n(\mathfrak{A}), M_n(\mathfrak{A})^*) \to \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$ is an isomorphism.

We want to pursue this further for more general Morita contexts. First we need to precisionize right exactness of the tensor functor:

2.8 Lemma. *Consider a short complex in* $\mathfrak{A}\mathbf{mod}$

$$X \xrightarrow{i} Y \xrightarrow{q} Z \to \{0\}$$

with $\mathrm{cl}(i(X)) = \ker(q)$ *and* $q(Y) = Z$ *(i.e. the dual complex* $X^* \xleftarrow{i^*} Y^* \xleftarrow{q^*} Z^* \leftarrow \{0\}$ *is exact). Then for any* $P \in \mathbf{mod}\mathfrak{A}$*:*

$$\mathrm{cl}((\mathbf{1}_P \otimes i)(P \underset{\mathfrak{A}}{\widehat{\otimes}} X)) = \ker(\mathbf{1}_P \otimes q) \text{ and } (\mathbf{1}_P \otimes q)(P \underset{\mathfrak{A}}{\widehat{\otimes}} Y) = (P \underset{\mathfrak{A}}{\widehat{\otimes}} Z).$$

Proof. The proof of [G1, Lemma 3.1] works verbatimly.

We use this to weaken the defining properties of Morita equivalence:

2.9 Lemma. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be any Banach algebras, and let* $P \in \mathfrak{A}\mathbf{mod}\mathfrak{B}$ *and* $Q \in \mathfrak{B}\mathbf{mod}\mathfrak{A}$*. Suppose that*
(1) $\mathfrak{A}$ *is self-induced;*
(2) *there are pairings*

$$\{\ ,\ \}\colon P \times Q \to \mathfrak{A}$$
$$[\ ,\ ]\colon Q \times P \to \mathfrak{B}$$

satisfying the compatibility conditions (§)*;*
(3) $\{\ ,\ \}$ *implements an epimorphism;*
(4) $\mathrm{cl}(\mathfrak{A}.P) = P$*.*

Then $\{\ ,\ \}$ *actually implements an isomorphism.*

Proof. Let $q\colon P \underset{\mathfrak{B}}{\widehat{\otimes}} Q \to \mathfrak{A}$ be the epimorphism implemented by $\{\ ,\ \}$ and set $K = \ker q$, so that we have a short exact sequence

$$\{0\} \to K \xrightarrow{i} P \underset{\mathfrak{B}}{\widehat{\otimes}} Q \xrightarrow{q} \mathfrak{A} \to \{0\}.$$

We want to show that $K = \{0\}$. Consider

$$\begin{array}{ccccccc}
\mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} K & \xrightarrow{\mathbf{1}_{\mathfrak{A}} \underset{\mathfrak{A}}{\otimes} i} & \mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} P \underset{\mathfrak{B}}{\widehat{\otimes}} Q & \xrightarrow{\mathbf{1}_{\mathfrak{A}} \underset{\mathfrak{A}}{\otimes} q} & \mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} \mathfrak{A} & \longrightarrow & \{0\} \\
{\scriptstyle \mu_1}\downarrow & & {\scriptstyle \mu_2}\downarrow & & {\scriptstyle \mu_3}\downarrow & & \\
K & \xrightarrow{i} & P \underset{\mathfrak{B}}{\widehat{\otimes}} Q & \xrightarrow{q} & \mathfrak{A} & \longrightarrow & \{0\}
\end{array}$$

where we for brevity have set $\mu_1 = \mu_K,\ \mu_2 = \mu_{P \underset{\mathfrak{B}}{\widehat{\otimes}} Q}$ and $\mu_3 = \mu_{\mathfrak{A}}$. By assumption μ_3 is an isomorphism and μ_2 has dense range. We show that μ_1 has dense range. Let $k \in K$. Since μ_2 has dense range, there is a sequence $t_n \in \mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} P \underset{\mathfrak{B}}{\widehat{\otimes}} Q$ so that $\lim \mu_2(t_n) = i(k)$. Hence $q(\mu_2(t_n)) \to q(i(k)) = 0$, so $\mu_3^{-1} \circ q \circ \mu_2(t_n) = \mathbf{1}_{\mathfrak{A}} \underset{\mathfrak{A}}{\otimes} q(t_n) \to 0$. By Lemma 2.8 and the open mapping theorem there is a sequence $u_n \in \mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} K$ so that $t_n - \mathbf{1}_{\mathfrak{A}} \underset{\mathfrak{A}}{\otimes} i(u_n) \to 0$. But then $\lim \mu_2(t_n - \mathbf{1}_{\mathfrak{A}} \underset{\mathfrak{A}}{\otimes} i(u_n)) \to 0$, i.e. $\lim \mu_2(\mathbf{1}_{\mathfrak{A}} \underset{\mathfrak{A}}{\otimes} i(u_n)) = i(k) \to 0$ or equivalentlly $\mu_1(u_n) \to k$.

Let $k = \sum p_n \underset{\mathfrak{B}}{\otimes} q_n$. For any $p \in P,\ q \in Q$ we have

$$\begin{aligned}
\{p, q\}k = \sum \{p, q\} p_n \underset{\mathfrak{B}}{\otimes} q_n &= \sum p[q, p_n] \underset{\mathfrak{B}}{\otimes} q_n \\
&= \sum p \underset{\mathfrak{B}}{\otimes} [q, p_n] q_n = \sum p \underset{\mathfrak{B}}{\otimes} q\{p_n, q_n\} = 0
\end{aligned}$$

so by (3) $\mathfrak{A}.K = \{0\}$ and we are done.

2.10 Theorem. *Let $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, P, Q, \dots)$ be a Morita context of self-induced Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that the pairings both implement epimorphisms.*
(1) With ${}_{\mathfrak{A}}P_{\mathfrak{B}} = \mathfrak{A}\widehat{\underset{\mathfrak{A}}{\otimes}}P\widehat{\underset{\mathfrak{B}}{\otimes}}\mathfrak{B}$ and ${}_{\mathfrak{B}}Q_{\mathfrak{A}} = \mathfrak{B}\widehat{\underset{\mathfrak{B}}{\otimes}}Q\widehat{\underset{\mathfrak{A}}{\otimes}}\mathfrak{A}$ we have a full Morita context $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, {}_{\mathfrak{A}}P_{\mathfrak{B}}, {}_{\mathfrak{B}}Q_{\mathfrak{A}}, \dots)$. In particular $\mathfrak{A} \sim \mathfrak{B}$
(2) Let $\mathcal{M}$ be the Banach algebra associated with $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, {}_{\mathfrak{A}}P_{\mathfrak{B}}, {}_{\mathfrak{B}}Q_{\mathfrak{A}}, \dots)$. Then $\mathcal{M}$ is self-induced and $\mathcal{M} \sim \mathfrak{A} \sim \mathfrak{B}$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
P\widehat{\underset{\mathfrak{B}}{\otimes}}Q\widehat{\underset{\mathfrak{A}}{\otimes}}P\widehat{\underset{\mathfrak{B}}{\otimes}}Q \arrow[r, "1"] \arrow[d, "2"'] & \mathfrak{A}\widehat{\underset{\mathfrak{A}}{\otimes}}\mathfrak{A} \arrow[d, "3"] \\
P\widehat{\underset{\mathfrak{B}}{\otimes}}\mathfrak{B}\widehat{\underset{\mathfrak{B}}{\otimes}}Q \arrow[r, "4"] & \mathfrak{A}
\end{tikzcd}$$

Since 1, 2, and 3 are epimorphism, 4 is also an epimorphism, so we have epimorphisms

$$(\mathfrak{A}\widehat{\underset{\mathfrak{A}}{\otimes}}P\widehat{\underset{\mathfrak{B}}{\otimes}}\mathfrak{B})\widehat{\underset{\mathfrak{B}}{\otimes}}(\mathfrak{B}\widehat{\underset{\mathfrak{B}}{\otimes}}Q\widehat{\underset{\mathfrak{A}}{\otimes}}\mathfrak{A}) \to \mathfrak{A}$$
$$(\mathfrak{B}\widehat{\underset{\mathfrak{B}}{\otimes}}Q\widehat{\underset{\mathfrak{A}}{\otimes}}\mathfrak{A})\widehat{\underset{\mathfrak{A}}{\otimes}}(\mathfrak{A}\widehat{\underset{\mathfrak{A}}{\otimes}}P\widehat{\underset{\mathfrak{B}}{\otimes}}\mathfrak{B}) \to \mathfrak{B}$$

It follows that ${}_{\mathfrak{A}}P_{\mathfrak{B}}$ and ${}_{\mathfrak{B}}Q_{\mathfrak{A}}$ satisfy the hypotheses of Lemma, thereby proving (1).

Thus we may assume that we have a full Morita context $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, P, Q, \{\ ,\ \}, [\ ,\ ])$ with associated Banach algebra

$$\mathcal{M} = \begin{pmatrix} \mathfrak{A} & P \\ Q & \mathfrak{B} \end{pmatrix},$$

by replacing the bimodules P and Q by the induced bimodules ${}_{\mathfrak{A}}P_{\mathfrak{B}}$ and ${}_{\mathfrak{B}}Q_{\mathfrak{A}}$ if necessary. Define $\mathfrak{P} = (\mathfrak{A}\ P)$ and $\mathfrak{Q} = \begin{pmatrix} \mathfrak{A} \\ Q \end{pmatrix}$. Then $\mathfrak{P}$ and $\mathfrak{Q}$ are both $\mathfrak{A}$- and $\mathcal{M}$- induced, and by means of matrix multiplication we have epimorphic compatible pairings. In order to finish by appealing to Lemma 2.9 we must show that $\mathcal{M}$ is self-induced. It is convenient to write $\mathcal{M} = \begin{pmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{pmatrix}$. Let $\phi \colon \mathcal{M}\times\mathcal{M} \to \mathbb{C}$ be an $\mathcal{M}$-balanced bilinear functional. Using the isomorphisms $\mathfrak{A}_{ij} \widehat{\underset{\mathfrak{A}_{jj}}{\otimes}} \mathfrak{A}_{jk} \cong \mathfrak{A}_{ik},\ i, j, k = 1, 2$ one finds linear functionals $\varphi_{ij} \colon \mathfrak{A}_{ij} \to \mathbb{C}$ such that with $\tilde{\phi} = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ one has $\phi((\mathfrak{a}_{ij}), (\alpha_{ij})) = \tilde{\phi}((\mathfrak{a}_{ij})(\alpha_{ij}))$. This is a routine but rather lengthy exercise along the lines of the proof of Proposition 2.5 and is therefore omitted. It follows that $\mathcal{M}$ has the universal property of $\mathcal{M}\widehat{\underset{\mathcal{M}}{\otimes}}\mathcal{M}$.

Remark. We cannot conclude the reverse to (2). Let X be a Banach space such that $\mathcal{A}(X)$ is not self-induced. Since $\mathcal{A}(X \oplus C_p)$ is weakly amenable ([B1, Corollary 3.3] and $X \oplus C_p$ is decomposable, then $\mathcal{A}(X \oplus C_p)$ is self-induced, ([G3, Theorem 2.9].

In order to compare cohomology of Morita equivalent Banach algebras we shall as previously exploit the double complex of Waldhausen [DI]. Let $\mathfrak{M}(\mathfrak{A},\mathfrak{B},P,Q,\dots)$ be a full Morita context of self-induced Banach algebras. The complex on the first axis of the Waldhausen bi-complex is the shifted Hochschild complex $C_{\bullet-1}(\mathfrak{A},\mathfrak{A})$ and on the second axis the shifted Hochschild complex $C_{\bullet-1}(\mathfrak{B},\mathfrak{B})$ The n'th column for $n\geq 1$ is the complex $P\underset{\mathfrak{B}}{\widehat{\otimes}}C_\bullet(\mathfrak{B},Q\widehat{\otimes}\mathcal{A}^{\widehat{\otimes}(n-1)})$ where $C_\bullet(\mathfrak{B},Q\widehat{\otimes}\mathfrak{A}^{\widehat{\otimes}(n-1)})$ is the normalized bar resolution of the left $\mathfrak{B}$-module $Q\widehat{\otimes}\mathcal{A}^{\widehat{\otimes}(n-1)}$. Similarly, the m'th row is the complex $Q\underset{\mathcal{A}}{\widehat{\otimes}}C_\bullet(\mathfrak{A},P\widehat{\otimes}\mathfrak{B}^{\widehat{\otimes}(m-1)})$. For details, see [G2].

The cohomology groups $\mathcal{H}^n((\cdot),(\cdot)^*)$ are related to the dual bi-cocomplex:

$$\begin{tikzcd}
(3) & (\mathfrak{B}\widehat{\otimes}\mathfrak{B}\widehat{\otimes}\mathfrak{B})^* \arrow[r] & {} \\
(2) & (\mathfrak{B}\widehat{\otimes}\mathfrak{B})^* \arrow[r] \arrow[u] & (P\widehat{\otimes}\mathfrak{B}\widehat{\otimes}Q)^* \arrow[r] \arrow[u] & {} \\
(1) & \mathfrak{B}^* \arrow[r] \arrow[u] & (P\widehat{\otimes}Q)^* \arrow[r] \arrow[u] & (P\widehat{\otimes}Q\widehat{\otimes}\mathfrak{A})^* \arrow[r] \arrow[u] & {} \\
 & 0 \arrow[r] \arrow[u] & \mathfrak{A}^* \arrow[r] \arrow[u] & (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^* \arrow[r] \arrow[u] & (\mathfrak{A}\widehat{\otimes}\mathfrak{A}\widehat{\otimes}\mathfrak{A})^* \arrow[u] \\
 & & (1) & (2) & (3)
\end{tikzcd}$$

Denote the E_p-terms for the spectral sequence whose E_2-terms are obtained by taking first horizontal and then vertical cohomology by ${}^IE_p^{(ij)}$ and the E_p-terms for the spectral sequence whose E_2-terms are obtained by taking first vertical and then horizontal cohomology by ${}^{II}E_p^{(ij)}$. We compute

$$\begin{aligned}
{}^IE_1^{(i0)} &= \mathcal{H}^{i-1}(\mathfrak{A},\mathfrak{A}^*) \qquad {}^IE_1^{(ij)} = \{0\}, \quad i=0,1;\ j\geq 1\\
{}^{II}E_1^{(0j)} &= \mathcal{H}^{j-1}(\mathfrak{B},\mathfrak{B}^*) \quad {}^{II}E_1^{(ij)} = \{0\}, \quad j=0,1;\ i\geq 1
\end{aligned}$$

where the terms on the axes follow from the definition of Hochschild cohomology and the others from the Morita context being full and the definition of the tensor products $\underset{\mathfrak{A}}{\widehat{\otimes}}$ and $\underset{\mathfrak{B}}{\widehat{\otimes}}$.

Thus ${}^IE_p^{(20)}$ and ${}^{II}E_p^{(02)}$ stabilize at $p=2$ and ${}^IE_1^{(02)} ={}^I E_1^{(11)} = {}^IE_1^{(20)} = {}^{II}E_1^{(20)} = {}^{II}E_1^{(11)} = {}^{II}E_1^{(02)} = \{0\}$. It follows that ${}^IE_2^{(20)} \cong {}^{II}E_2^{(02)}$. We describe these E_2-terms in a definition.

2.11 Definition. Let $\mathfrak{M}(\mathfrak{A},\mathfrak{B},P,Q,\{\ ,\ \},[\ ,\ ])$ be Morita context of self-induced Banach algebras, and let $D\colon\mathfrak{A}\to\mathfrak{A}^*$ be a derivation. Then $D\in\mathcal{Z}^1_{\mathfrak{M}}(\mathfrak{A},\mathfrak{A}^*)$, if the tri-linear form $(p,q,a)\mapsto\langle\{p,q\},D(a)\rangle$ is expressed as

$$\langle\{p,q\},D(a)\rangle = \psi(ap,q)-\psi(p,qa), \quad p\in P, q\in Q, a\in\mathfrak{A}$$

for some bilinear form $\psi: P \times Q \to \mathbb{C}$. We define a subgroup of $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$ by

$$\mathcal{H}^1_{\mathfrak{M}}(\mathfrak{A}, \mathfrak{A}^*) = \mathcal{Z}^1_{\mathfrak{M}}(\mathfrak{A}, \mathfrak{A}^*)/\mathcal{B}^1(\mathfrak{A}, \mathfrak{A}^*),$$

where $\mathcal{B}^1(\mathfrak{A}, \mathfrak{A}^*)$ is the space of inner derivations. The subgroup $\mathcal{H}^1_{\mathfrak{M}}(\mathfrak{B}, \mathfrak{B}^*)$ of $\mathcal{H}^1(\mathfrak{B}, \mathfrak{B}^*)$ is defined analogously.

2.12 Examples. We give some instances, where $\mathcal{H}^1_{\mathfrak{M}}(\cdot, \cdot^*) = \mathcal{H}^1(\cdot, \cdot^*)$
(1) Obviously (but importantly), if $\mathfrak{A}$ is weakly amenable, then $\mathcal{H}^1_{\mathfrak{M}}(\mathfrak{A}, \mathfrak{A}^*) = \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$
(2) Suppose that $\mathfrak{B}$ is unital and that $[\ ,\ ]$ implements an epimorphism $Q \widehat{\underset{\mathfrak{A}}{\otimes}} P \mapsto \mathfrak{B}$. Then we may write $\mathbf{1}_{\mathfrak{B}} = \sum [q_n, q_n]$ and any derivation D has $\langle \{p, q\}, D(a) \rangle = \psi(ap, q) - \psi(p, qa)$ with

$$\psi(p, q) = \sum \langle \{p_n, q\}, D(\{p, q_n\} \rangle.$$

(3) Let $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, P, Q, \dots)$ be a Morita context and let $\mathcal{M} = \begin{pmatrix} \mathfrak{A} & P \\ Q & \mathfrak{B} \end{pmatrix}$. Let $\mathfrak{N}(\mathcal{M}, \mathfrak{A}, \begin{pmatrix} \mathfrak{A} \\ Q \end{pmatrix}, (\mathfrak{A}\ P), \dots)$ be the iterated Morita context with pairings defined by matrix multiplication. Then for any derivation $D: \mathcal{M} \to \mathcal{M}^*$

$$\langle \begin{pmatrix} a \\ q \end{pmatrix} (c\ p), D(\begin{pmatrix} a' & p' \\ q' & b' \end{pmatrix}) \rangle =$$
$$\langle \begin{pmatrix} a & 0 \\ q & 0 \end{pmatrix}, D(\begin{pmatrix} c & p \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & p' \\ q' & b' \end{pmatrix}) \rangle - \langle \begin{pmatrix} c & p \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & p' \\ q' & b' \end{pmatrix}, D(\begin{pmatrix} a & 0 \\ q & 0 \end{pmatrix}) \rangle$$

so that $\mathcal{H}^1_{\mathfrak{N}}(\mathcal{M}, \mathcal{M}^*) = \mathcal{H}^1(\mathcal{M}, \mathcal{M}^*)$

Morita invariance of 1st degree cohomology now reads

2. 13 Theorem. *Let $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, P, Q, \dots)$ be a full Morita context of self-induced Banach algebras. Then*

$$\mathcal{H}^1_{\mathfrak{M}}(\mathfrak{A}, \mathfrak{A}^*) \cong \mathcal{H}^1_{\mathfrak{M}}(\mathfrak{B}, \mathfrak{B}^*).$$

3. Applications to Banach algebras of approximable operators.

In [G1] we showed that for Banach algebras of approximable operators on Banach spaces, whose dual have the bounded approximation property, Morita equivalence is determined by approximate factorization. Specifically, $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are Morita equivalent if and only if the multiplications $\mathcal{A}(Y, X) \widehat{\otimes} \mathcal{A}(X, Y) \to \mathcal{A}(X)$ and $\mathcal{A}(X, Y) \widehat{\otimes} \mathcal{A}(Y, X) \to \mathcal{A}(Y)$ both are surjective. This can now be improved to the much wider class for which $\mathcal{A}(X)$ is self-induced. As in [G1] the key is to find the irreducible modules.

3.1 Lemma. *X is the only irreducible left module over $\mathcal{A}(X)$.*

Proof. Let L be a closed left ideal of $\mathcal{A}(X)$ and define

$$\Psi(L) = \bigcup_{A \in L} A^*(X^*)\ .$$

Then $\Psi(L)$ is a closed subspace of X^*: Let $\xi, \eta \in X^*$ and $A, B \in L$. Choose $< x, x^* >= 1$. Then

$$\big((x \otimes \xi)A + (x \otimes \eta)B\big)^* x^* = A^*\xi + B^*\eta\ ,$$

so $\Psi(L) + \Psi(L) \subseteq \Psi(L)$. Clearly $\mathbb{C}\Psi(L) \subseteq \Psi(L)$. To see that $\Psi(L)$ is closed, let $A_n \in L$, $\xi_n \in X^*$ and suppose $A_n^*\xi_n \to \xi$. Then $(x \otimes \xi_n)A_n \to x \otimes \xi \in L$, so $\xi = (x \otimes \xi)^* x^* \in \Psi(L)$, i.e. $\Psi(L)$ is closed.

Now suppose that $\Psi(L) = X^*$ and let $x \otimes \eta$ be an arbitrary rank-1 operator. Since $\Psi(L) = X^*$, we may choose $A \in L$ and $x^* \in X^*$ so that $A^*x^* = \eta$. Then $x \otimes \eta = (x \otimes x^*)A \in L$. Consequently L contains all finite rank operators, that is, $L = \mathcal{A}(X)$.

It is clear that X is an irreducible $\mathcal{A}(X)$-module. Let L be a maximal modular left ideal of $\mathcal{A}(X)$. By the preceeding paragraph we may choose $\xi \in X^* \setminus \Psi(L)$. Let $Q\colon \mathcal{A}(X) \to \mathcal{A}(X)/L$ be the canonical map. Then

$$x \mapsto Q(x \otimes \xi)\colon X \to \mathcal{A}(X)/L$$

is a non-zero (bounded) module map between irreducible modules and hence an isomorphism.

3.2 Theorem. *Suppose that $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are self-induced. Then $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are Morita equivalent, if and only if multiplication implements surjections*

$$\begin{aligned} &\mathcal{A}(X,Y)\widehat{\otimes}\mathcal{A}(Y,X) \to \mathcal{A}(Y) \\ &\mathcal{A}(Y,X)\widehat{\otimes}\mathcal{A}(X,Y) \to \mathcal{A}(X). \end{aligned}$$

Proof. The proof is along the same lines as in [G1, Theorem 7.5] so we give only a sketch. For brevity we set $\mathfrak{A} = \mathcal{A}(X)$ and $\mathfrak{B} = \mathcal{A}(Y)$. Let the Morita equivalence be implemented by a full Morita context $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, P, Q, \dots)$, and let $\Phi\colon P\underset{\mathfrak{B}}{\widehat{\otimes}}Q \to \mathfrak{A}$ be the corresponding isomorphism. Since irreducibility of modules is Morita invariant there is a bounded $\mathfrak{B}$-module isomorphism $\kappa\colon Q\underset{\mathfrak{A}}{\widehat{\otimes}}X \to Y$. For each $q \in Q$ we define an operator $X \to Y$ by $x \mapsto \kappa(q \underset{\mathfrak{A}}{\otimes} x)$, and for each $p \in P$ we define an operator $Y \to X$ by

$$y \mapsto \mu_X \circ (\Phi \underset{\mathfrak{A}}{\otimes} \mathbf{1}_X) \circ (\mathbf{1}_P \underset{\mathfrak{B}}{\otimes} \kappa^{-1})(p \underset{\mathfrak{B}}{\otimes} y).$$

It follows that each $S \in \mathcal{A}(X)$ has a factorization

$$S = \sum_n U_n V_n,\ U_n \in \mathcal{B}(Y,X), V_n \in \mathcal{B}(X,Y).$$

Since $\mathcal{A}(X)$ is self-induced we can actually obtain $U_n \in \mathcal{A}(Y,X), V_n \in \mathcal{A}(X,Y)$.

Conversely, suppose that multiplications are surjective. Then $P = \mathcal{A}(Y,X)$ and $Q = \mathcal{A}(X,Y)$ with pairings given by composition of operators clearly satisfy the conditions of Lemma , so that the Morita context $\mathfrak{M}(\mathfrak{A}, \mathfrak{B}, P, Q, \dots)$ is full.

We shall now investigate Hochschild cohomology groups related to certain tensor products, of which the vector valued L_p-spaces are prototypical. First a definition to describe the situation

3.3 Definition. Let X and Y be two Banach spaces. We say that Y *is strongly finitely represented in* X if there is $C>0$ such that for each finite dimensional subspace $S\subseteq Y$ there are a finite dimensional subspace $S\subseteq E$ and linear maps $T_E\colon E\to X$, $\iota_E\colon X\to Y$ such that $\iota_E T_E$ is the inclusion $E\longrightarrow Y$ and $\|T_E\|\|\iota_E\|\le C$.

3.4 Examples. The $\mathcal{L}_p$-spaces are defined to be the Banach spaces which are finitely represented in ℓ_p in the usual (weaker) sense of finite representability. However they satisfy the stronger condition of Definition 3.3, see [L&R, Theorem III]. Any Banach space is strongly finitely represented in the C_p-spaces of W.B. Johnson.

We now define the tensor products that are compliant with finite representability.

3.5 Definition. Let Y and Z be Banach spaces, let $\|\cdot\|$ be a reasonable cross-norm on $Y\otimes Z$ with completion $Y\bar{\otimes}Z$. For any complemented subspace $X\subseteq Y$, set $X\bar{\otimes}Z=\mathrm{cl}(X\otimes Z)$. We say that $Y\bar{\otimes}Z$ is *compliable* if for each complemented subspace $X\subseteq Y$ the map $x\otimes z\mapsto Tx\otimes z$ defines a linear map $\mathcal{B}(X,Y)\mapsto\mathcal{B}(X\bar{\otimes}Z,Y\bar{\otimes}Z)$ of bound ≤ 1.

3.6 Examples. (1) Let X be a complemented subspace of Y with complementation constant $C\ge 1$ and let $Y\bar{\otimes}Z$ be a compliable tensor product. Then $X\bar{\otimes}Z$ is compliable tensor product and a complemented subspace of $Y\bar{\otimes}Z$ with complementation constant C.
(2) Let X be any Banach space, and let (ν,Ω,Σ) be a measure space. Then the Banach spaces of p-Bocher integrable X-valued functions $L_p(\nu,X)$, $1\le p\le\infty$ are compliable tensor products.

3.7 Proposition. *Suppose that Y is strongly finitely represented in X and that $Y\cong Y\oplus X$. Let Z be any Banach space and let $Y\bar{\otimes}Z$ be a compliable tensor product. If $\mathcal{A}(Y\bar{\otimes}Z)$ is self-induced, then*

(1) *$\mathcal{A}(X\bar{\otimes}Z)$ is self-induced;*
(2) *$\mathcal{A}(Y\bar{\otimes}Z)\sim\mathcal{A}(X\bar{\otimes}Z)$*
(3) *$\mathcal{H}^1(\mathcal{A}(Y\bar{\otimes}Z),\mathcal{A}(Y\bar{\otimes}Z)^*)\cong\mathcal{H}^1_{\mathfrak{M}}(\mathcal{A}(X\bar{\otimes}Z),\mathcal{A}(X\bar{\otimes}Z)^*)$, where $\mathfrak{M}$ is the canonical Morita context $\mathfrak{M}(\mathcal{A}(Y\bar{\otimes}Z\oplus X\bar{\otimes}Z),\mathcal{A}(X\bar{\otimes}Z),\dots)$.*

Proof. Let for brevity $\mathfrak{Y}=Y\bar{\otimes}Z$ and $\mathfrak{X}=X\bar{\otimes}Z$. Then $\mathfrak{Y}\cong\mathfrak{Y}\oplus\mathfrak{X}$ by compliance, so that $\mathcal{A}(\mathfrak{X})$ may be viewed as a corner of $\mathcal{A}(\mathfrak{Y})$. Since we have assumed that $\mathcal{A}(\mathfrak{Y})$ is self-induced, any operator $A\in\mathcal{A}(\mathfrak{X})$ has an approximate factorization $A=\sum U_nV_n$ with $U_n\in\mathcal{F}(\mathfrak{Y},\mathfrak{X})$, $V_n\in\mathcal{F}(\mathfrak{X},\mathfrak{Y})$. By the defining properties of strongly finite representability and of compliance, for each $n\in\mathbb{N}$ there is a finite dimensional subspace $E_n\subseteq Y$ so that $(\iota_{E_n}\bar{\otimes}\mathbf{1}_Z)(T_{E_n}\bar{\otimes}\mathbf{1}_Z)V_n=V_n$. It follows that the multiplication $\mathcal{A}(\mathfrak{X})\underset{\mathcal{A}(\mathfrak{X})}{\widehat{\otimes}}\mathcal{A}(\mathfrak{X})\to\mathcal{A}(\mathfrak{X})$ is surjective.

To show that it is injective, let ϕ be a bounded $\mathcal{A}(\mathfrak{X})$-balanced bilinear functional on $\mathcal{A}(\mathfrak{X})$. According to the decomposition $\mathfrak{Y}\cong\mathfrak{Y}\oplus\mathfrak{X}$ we may consider operators in $\mathcal{A}(\mathfrak{Y})$ as matrices

$$\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix},\quad A_{11}\in\mathcal{A}(\mathfrak{Y}),A_{12}\in\mathcal{A}(\mathfrak{X},\mathfrak{Y}),A_{21}\in\mathcal{A}(\mathfrak{Y},\mathfrak{X}),A_{22}\in\mathcal{A}(\mathfrak{X}).$$

Let $(A_{ij}),(B_{ij})\in\mathcal{F}(\mathfrak{Y})$ and define

$$\widetilde{\phi}((A_{ij}),(B_{ij}))=$$
$$\lim_{E\to\infty}\phi((T_E\bar{\otimes}\mathbf{1}_Z\ \mathbf{1}_{\mathfrak{X}})(A_{ij})\begin{pmatrix}\iota_E\bar{\otimes}\mathbf{1}_Z\\\mathbf{1}_{\mathfrak{X}}\end{pmatrix},(T_E\bar{\otimes}\mathbf{1}_Z\ \mathbf{1}_{\mathfrak{X}})(B_{ij})\begin{pmatrix}\iota_E\bar{\otimes}\mathbf{1}_Z\\\mathbf{1}_{\mathfrak{X}}\end{pmatrix}),$$

where the ordering is by inclusion of E's (and formally $(T_E\bar{\otimes}\mathbf{1}_Z)A_{ij} = 0$ if the range of A_{ij} is not included in $E\bar{\otimes}Z$.

This is easily seen to define a bounded $\mathcal{A}(\mathfrak{Y})$-balanced bilinear functional on $\mathcal{A}(\mathfrak{Y})$ extending ϕ from the corner. Now suppose that $\sum U_nV_n = 0$ $,U_n, V_n \in \mathcal{A}(\mathfrak{X})$. Then $\sum \begin{pmatrix} 0 & 0 \\ 0 & U_n \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & V_n \end{pmatrix} = 0$. Since $\mathcal{A}(\mathfrak{Y})$ is self-induced, we have that $\sum \widetilde{\phi}(\begin{pmatrix} 0 & 0 \\ 0 & U_n \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & V_n \end{pmatrix}) = 0$. But then $\sum \phi(U_n, V_n) = 0$, i.e. multiplication $\mathcal{A}(\mathfrak{X}) \underset{\mathcal{A}(\mathfrak{X})}{\widehat{\otimes}} \mathcal{A}(\mathfrak{X}) \to \mathcal{A}(\mathfrak{X})$ is injective.

To show (2) one may argue similarly or just appeal to Theorem 2.10(1).

The statement (3) follows from Theorem 2.13 by noting that if $\mathfrak{M}$ is the canonical Morita context of $\mathcal{A}(\mathfrak{Y} \oplus \mathfrak{X}) \sim \mathcal{A}(\mathfrak{X})$, then $\mathcal{H}^1_{\mathfrak{M}}(\mathcal{A}(\mathfrak{Y} \oplus \mathfrak{X}), \mathcal{A}(\mathfrak{Y} \oplus \mathfrak{X})^*) = \mathcal{H}^1(\mathcal{A}(\mathfrak{Y}\oplus\mathfrak{X}), \mathcal{A}(\mathfrak{Y}\oplus\mathfrak{X})^*)$, so that $\mathcal{H}^1(\mathcal{A}(\mathfrak{Y}), \mathcal{A}(\mathfrak{Y})^*) \cong \mathcal{H}^1(\mathcal{A}(\mathfrak{Y}\oplus\mathfrak{X}), \mathcal{A}(\mathfrak{Y}\oplus\mathfrak{X})^*) = \mathcal{H}^1_{\mathfrak{M}}(\mathcal{A}(\mathfrak{Y} \oplus \mathfrak{X}), \mathcal{A}(\mathfrak{Y} \oplus \mathfrak{X})^*) \cong \mathcal{H}^1_{\mathfrak{M}}(\mathcal{A}(\mathfrak{X}), \mathcal{A}(\mathfrak{X})^*)$.

In [G3] we proved that $\mathcal{A}(L_p(\nu, X))$ is weakly amenable for any infinite-dimensional $L_p(\nu)$ if X has the bounded approximation property. This can now be improved to

3.8 Corollary. *Let Y be an infinite-dimensional $\mathcal{L}_p$-space, $1 \le p < \infty$, and let $\mathfrak{Y} = Y\bar{\otimes}Z$ be a compliable tensor product. Then $\mathcal{A}(\mathfrak{Y})$ is weakly amenable, if and only if $\mathcal{A}(\mathfrak{Y})$ is self-induced. In particular for an infinite-dimensional $L_p(\nu)$, $\mathcal{A}(L_p(\nu, X))$ is weakly amenable, if and only if it is self-induced.*

Proof. By [G3, Theorem 2.9] being self-induced is necessary. The rest follows from noting that
- when $Y$is an $\mathcal{L}_p$-space, then $Y \cong Y \oplus \ell_p$ ([L&R, Theorem I]);
- $L_p(\nu, X)$ is (isometrically) isomorphic to a compliable tensor product $L_p(\nu)\bar{\otimes}X$;
- $\mathcal{A}(\ell_p(X))$ is weakly amenable, if and only it is self-induced, ([G3, Corollary 4.2]).

We finish the paper with some clarification concerning direct sums and self-inducedness. Recall that a derivation $D\colon \mathcal{A}(X) \to \mathcal{A}(X)^*$ has the form $\langle A, D(B)\rangle = \mathrm{tr}((AB - BA)^*T)$ for an appropriate $T \in \mathcal{B}(X^*)$ and that D is inner if and only if $T \in \mathcal{I}(X^*) + \mathbb{C}\mathbf{1}_{X^*}$.

Notation: For brevity we shall in the following use [**;**] for commutators (of matrices).

3.9 Lemma. *Let $D\colon \mathcal{A}(X \oplus Y) \to \mathcal{A}(X \oplus Y)^*$ be a derivation implemented by*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{B}(X^* \oplus Y^*).$$

If $\mathcal{A}(X) \underset{\mathcal{A}(X)}{\widehat{\otimes}} \mathcal{A}(Y, X) = \mathcal{A}(Y, X)$, then $T_{12} \in \mathcal{I}(Y^*, X^*)$. If $\mathcal{A}(X, Y) \underset{\mathcal{A}(X)}{\widehat{\otimes}} \mathcal{A}(X) = \mathcal{A}(X, Y)$, then $T_{21} \in \mathcal{I}(X^*, Y^*)$. If $\mathcal{A}(X \oplus Y)$ is self-induced, then $T_{12} \in \mathcal{I}(Y^*, X^*)$ and $T_{21} \in \mathcal{I}(X^*, Y^*)$.

Proof. For $A \in \mathcal{F}(X)$ and $P \in \mathcal{F}(Y,X)$ we have

$$\begin{aligned}
\|A\|\|P\|\|D\| &\geq |<\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, D(\begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix})>| \\
&= |\operatorname{tr}\left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}\right]^* \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix})| \\
&= |\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & (AP)^*T_{12} \end{pmatrix}| \\
&= |\operatorname{tr}((AP)^*T_{12})| \,.
\end{aligned}$$

Thus $(A,P) \mapsto \operatorname{tr}((AP)^*T_{12})$ defines a bounded, balanced bilinear form. Invoking the hypothesis we see that T_{12} defines a bounded linear form on $\mathcal{A}(Y,X)$, i.e. T_{12} is integral. The proof of the second statement follows analogously. Now suppose that $\mathcal{A}(X \oplus Y)$ is self-induced. This means that the four multiplications

$$\begin{aligned}
E_{11}&: \begin{pmatrix} \mathcal{A}(X) & \mathcal{A}(Y,X) \\ 0 & 0 \end{pmatrix} \underset{\mathcal{A}(X\oplus Y)}{\widehat{\otimes}} \begin{pmatrix} \mathcal{A}(X) & 0 \\ \mathcal{A}(X,Y) & 0 \end{pmatrix} \to \mathcal{A}(X) \\
E_{12}&: \begin{pmatrix} \mathcal{A}(X) & \mathcal{A}(Y,X) \\ 0 & 0 \end{pmatrix} \underset{\mathcal{A}(X\oplus Y)}{\widehat{\otimes}} \begin{pmatrix} 0 & \mathcal{A}(Y,X) \\ 0 & \mathcal{A}(Y) \end{pmatrix} \to \mathcal{A}(Y,X) \\
E_{21}&: \begin{pmatrix} 0 & 0 \\ \mathcal{A}(X,Y) & \mathcal{A}(Y) \end{pmatrix} \underset{\mathcal{A}(X\oplus Y)}{\widehat{\otimes}} \begin{pmatrix} \mathcal{A}(X) & 0 \\ \mathcal{A}(X,Y) & 0 \end{pmatrix} \to \mathcal{A}(X,Y) \\
E_{22}&: \begin{pmatrix} 0 & 0 \\ \mathcal{A}(X,Y) & \mathcal{A}(Y) \end{pmatrix} \underset{\mathcal{A}(X\oplus Y)}{\widehat{\otimes}} \begin{pmatrix} 0 & \mathcal{A}(Y,X) \\ 0 & \mathcal{A}(Y) \end{pmatrix} \to \mathcal{A}(Y)
\end{aligned}$$

are linear topological isomorphisms. For $A \in \mathcal{F}(X),\ P_1, P_2 \in \mathcal{F}(Y,X),\ B \in \mathcal{F}(Y)$ we get

$$\begin{aligned}
(\|A\| + \|P_1\|)(\|P_2\| + \|B\|)\|D\| &\geq |<\begin{pmatrix} A & P_1 \\ 0 & 0 \end{pmatrix}, D\begin{pmatrix} 0 & P_2 \\ 0 & B \end{pmatrix}>| \\
&= |\operatorname{tr}(\left[\begin{pmatrix} A & P_1 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & P_2 \\ 0 & B \end{pmatrix}\right]^* \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix})| \\
&= |\operatorname{tr}(\begin{pmatrix} 0 & 0 \\ (AP_2 + P_1B)^* & 0 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix})| \\
&= |\operatorname{tr}((AP_2 + P_1B)^*T_{12})|
\end{aligned}$$

Invoking the topological isomorphism E_{12} we see that T_{12} implements a bounded linear functional on $\mathcal{A}(Y,X)$ i.e. T_{12} is integral. Similarly, T_{21} is integral.

3.10 Theorem. *Let X and Y be infinite dimensional Banach spaces and assume that $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are WA. Then $\mathcal{A}(X \oplus Y)$ is WA if and only if $\mathcal{A}(X \oplus Y)$ is self-induced and* $\sup\{|\operatorname{tr} PQ| \mid P \in \mathcal{F}(Y,X)_1,\ Q \in \mathcal{F}(X,Y)_1\} = +\infty$.

Proof. Since $X \oplus Y$ is decomposable, self-inducedness is necessary ([G3, Theorem 2.9]). The other necessary condition is [B1, Proposition 4.1].

Now assume that $\mathcal{A}(X \oplus Y)$ is self-induced and that the supremum is infinite. Then, by the lemma, T_{12} and T_{21} are integral and since $\mathcal{A}(X)$ is WA, there is $\lambda \in \mathbb{C}$ so that $T_{11} + \lambda \mathbf{1}_{X^*} \in \mathcal{I}(X^*)$. By subtracting the inner derivation given by

$$\begin{pmatrix} T_{11} + \lambda \mathbf{1}_{X^*} & T_{12} \\ T_{21} & 0 \end{pmatrix}$$

we see, after adjusting by a multiple of $\mathbf{1}_{X^*\oplus Y^*}$ if necessary, that in order to show that $\mathcal{A}(X\oplus Y)$ is WA it suffices to look at derivations given by $T\in\mathcal{B}(X^*\oplus Y^*)$ of the form

$$T=\begin{pmatrix}0 & 0\\ 0 & T_{22}\end{pmatrix}.$$

Since $\mathcal{A}(Y)$ is WA, there is $\lambda\in\mathbb{C}$ so that $T_{22}+\lambda\mathbf{1}_{Y^*}\in\mathcal{I}(Y^*)$. If $\lambda\neq 0$, then

$$\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}_{Y^*}\end{pmatrix}$$

defines a bounded derivation. But then there is $K>0$ so that

$$|\operatorname{tr}(PQ)|=\left|\operatorname{tr}\left[\begin{pmatrix}0 & 0\\ Q & 0\end{pmatrix};\begin{pmatrix}0 & P\\ 0 & 0\end{pmatrix}\right]^*\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}_{Y^*}\end{pmatrix}\right|\leq K\|P\|\|Q\|$$

contrary to assumption. Hence $T_{22}\in\mathcal{I}(Y^*)$ and the derivation is inner as wanted.

Question. Theorem 6.10 of [G,J&W] states that if $\mathcal{A}(X\oplus Y)$ is amenable, then at least one of $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ is amenable. Can 'amenable' be replaced by 'weakly amenable'? Note that $\mathcal{A}(X\oplus C_p)$ is weakly amenable for any Banach space ([B1, corollary 3.3]), so that the conjecture is best possible.

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